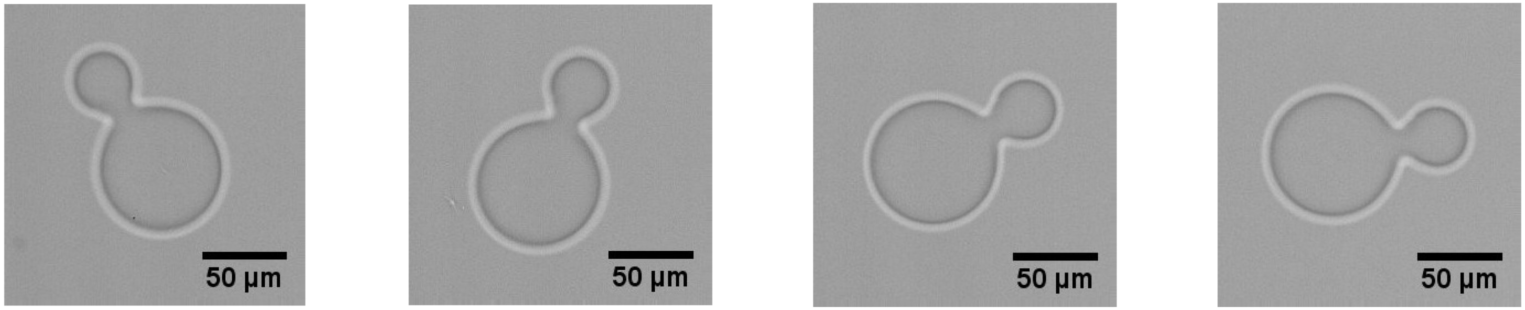
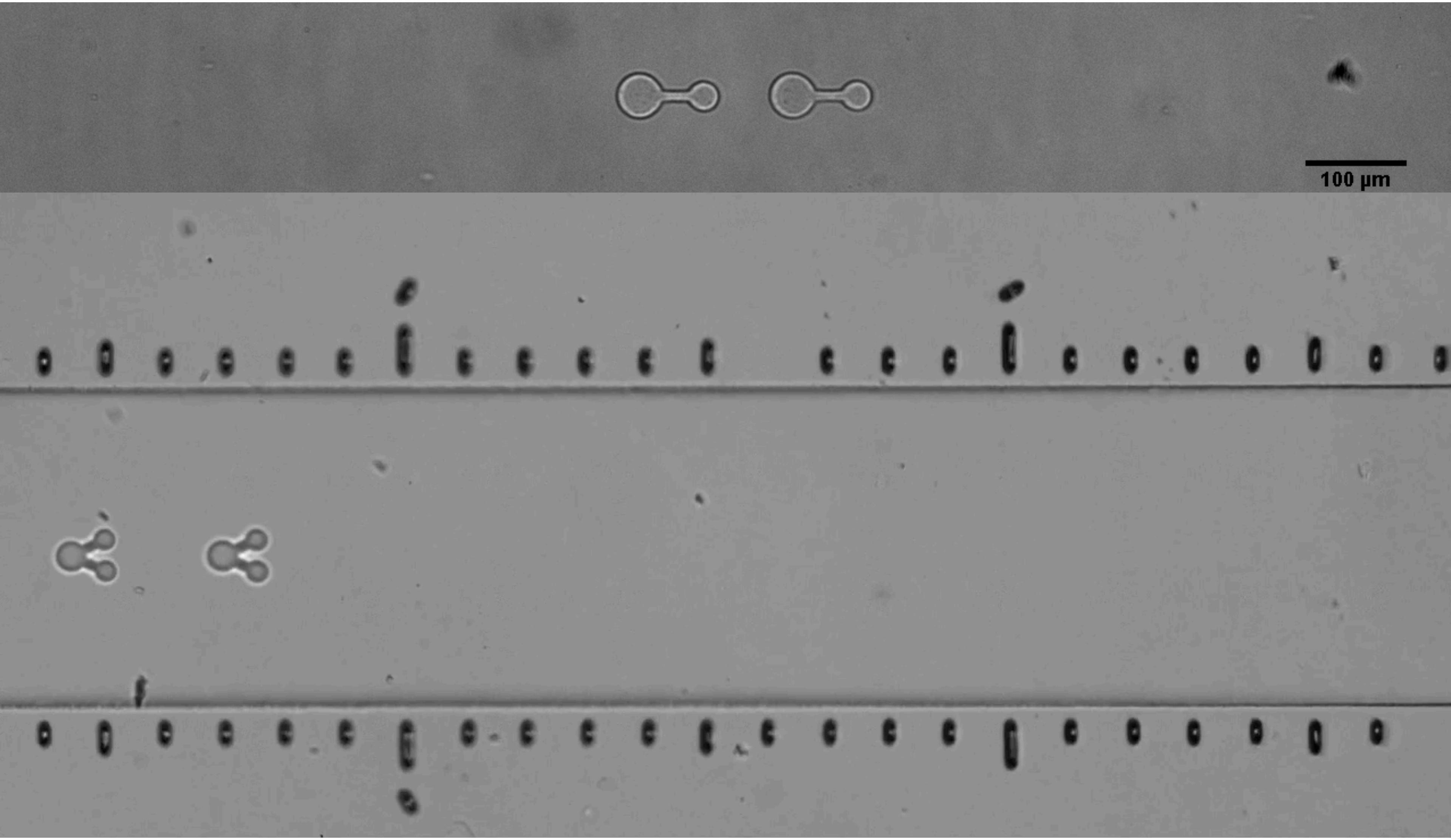


(a) $t/\tau = 1$ (b) $t/\tau = 2$ (c) $t/\tau = 3$ (d) $t/\tau = 4$

# A very short introduction of colloidal/Fluid dynamics

H.B.Eral
P&E TU Delft
www.erallab.com

# Model particles separated with flow



# Suggested reading, video and books

- Books:
  - E.Guazzelli & J.F.M Morris. “A Physical Introduction to Suspension Dynamics”
  - J.Happel & H.Brenner “Low Reynolds number hydrodynamics”
  - W.M. Deen “Analysis of Transport Phenomena” & “Introduction of Chemical Engineering Fluid Mechanics”
- Video:
  - G.I.Taylor “Low Reynolds number flows”
- Articles:
  - “Life at low Reynolds numbers” E.M.Purcell
  - Article: “Hydrodynamics at low Reynolds numbers” EJ Hinch

# Dimensional analysis

- Physical results **can not** depend on units that you choose.

Fundamental dimensions(n)

Mass [m]

Length [L]

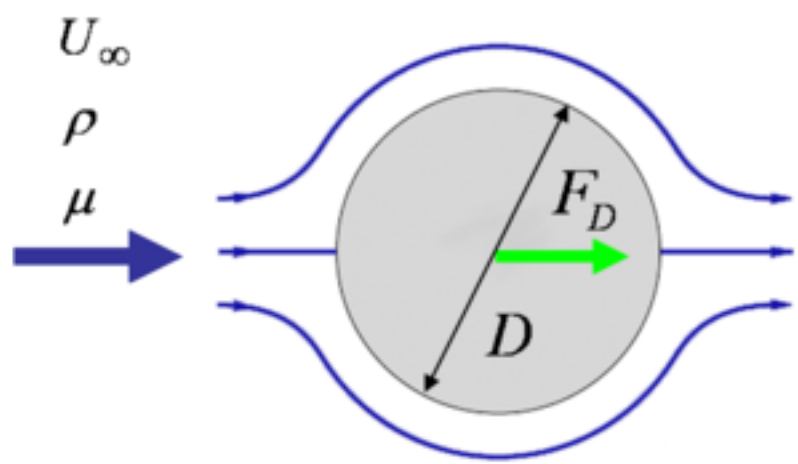
Time [t]

Derived units/parameter (p)

Force $[mL/t^2]$

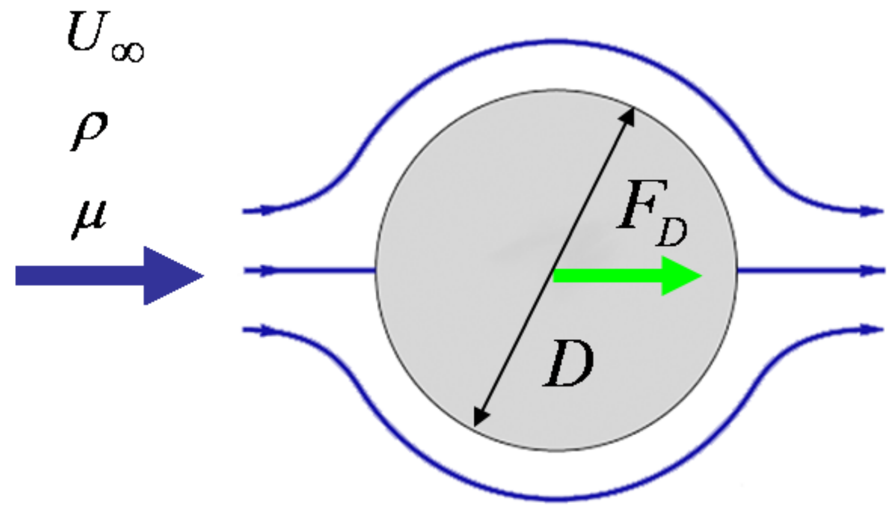
$Buckingham - Pi$

$\Pi = p - n$




Source:Wikipedia

## Buckingham-Pi theorem




- Drag on a sphere
  - Given $F_D = f(U, D, \mu, \rho)$
  - Objective: Find number of dimensionless groups

Parameters: $F_D, U, D, \mu, \rho$ i.e. p=5

Dimensions: Mass(M),Length(L),t i.e. d=3

Number of independent dimensionless groups, Π=p-d=5-3=2

$\Pi_1 = F_D, U, D, \rho$

$\Pi_2 = \mu, U, D, \rho$

Write the the parameters as a function of primary dimensions

$$F_D = [\frac{ML}{t^2}], U = [\frac{L}{t}], D = [L], \rho = [\frac{M}{L^3}], \mu = [\frac{ML}{t}]$$

$\Pi_1 = F_D, U, D, \rho$

$\Pi_2 = \mu, U, D, \rho$

Write the the parameters as a function of primary dimensions

$$F_D = [\frac{ML}{t^2}], U = [\frac{L}{t}], D = [L], \rho = [\frac{M}{L^3}], \mu = [\frac{M}{Lt}]$$

$$\Pi_1 = \left[\frac{ML}{t^2}\right], \left[\frac{L}{t}\right]^{a1}, L^{b1}, \left[\frac{M}{L^3}\right]^{c1}$$

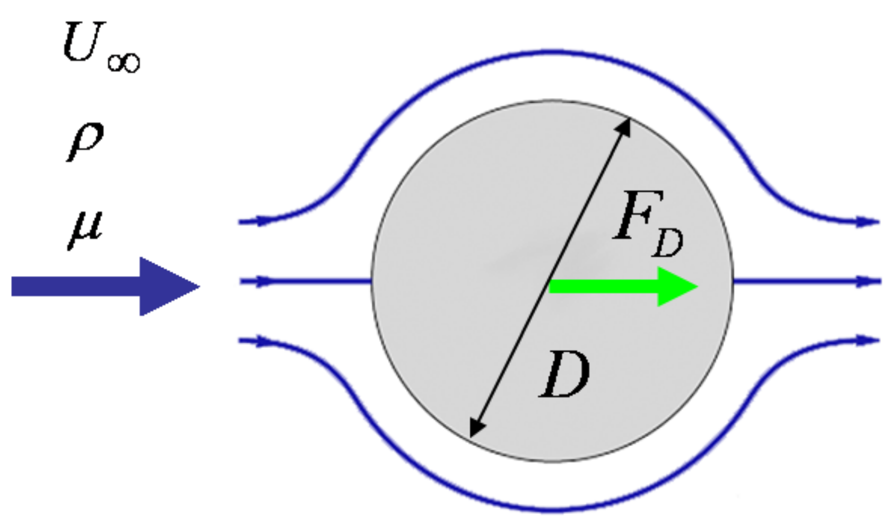
$[M]: 1 + c1 = 0$

$[L]: 1 + a1 + b1 - 3c1 = 0$

$[t]: -2 - a1 = 0$

c1=-1,a1=-2,b1=-2

$$\Pi_1 = FU^{-2}D^{-2}\rho^{-1} = \frac{F}{U^2 D^2 \rho} = C_D$$



$$\Pi_1 = F_D, U, D, \rho$$

$$\Pi_2 = \mu, U, D, \rho$$

Write the the parameters as a function of primary dimensions

$$F_D = [\frac{ML}{t^2}], U = [\frac{L}{t}], D = [L], \rho = [\frac{M}{L^3}], \mu = [\frac{M}{Lt}]$$

$$\Pi_2 = \left[\frac{M}{Lt}\right], \left[\frac{L}{t}\right]^{a1}, L^{b1}, \left[\frac{M}{L^3}\right]^{c1}$$

$$[M]: 1 + c1 = 0$$

$$[L]: -1 + a1 + b1 - 3c1 = 0$$

$$[t]: -1 - a1 = 0$$

a1=-1,c1=-1,b1=-1

$$\Pi_2 = \mu U^{-1} D^{-1} \rho^{-1} = \mathrm{Re}^{-1}$$

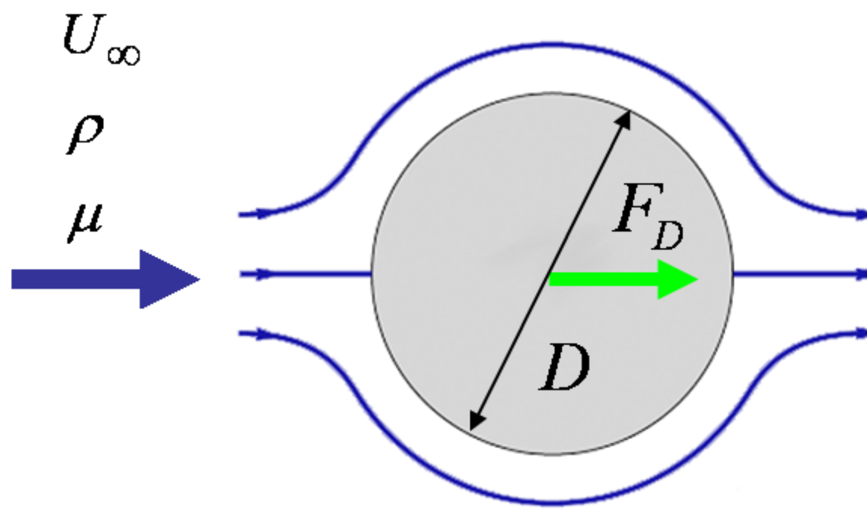
$$\mathrm{Re} = \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{\rho U^2}{\frac{\mu U}{L}} = \frac{\rho U L}{\mu}$$

$$\Pi_1 = FU^{-2}D^{-2}\mu^{-1} = \frac{F}{U^2 D^2 \rho}$$

$$\Pi_2 = \mu U^{-1} D^{-1} \rho^{-1} = \mathrm{Re}^{-1}$$

$$\Pi_1 = f(\Pi_2)$$

$$\boxed{\frac{F}{U^2 D^2 \mu} = f(\mathrm{Re}^{-1})}$$



# Physical understanding

- To find drag all we have to do is to vary Re and measure F.
- We do not have to vary all four parameters in Re just varying one while keeping the rest constant is sufficient i.e. n^4 experiments reduced to n.

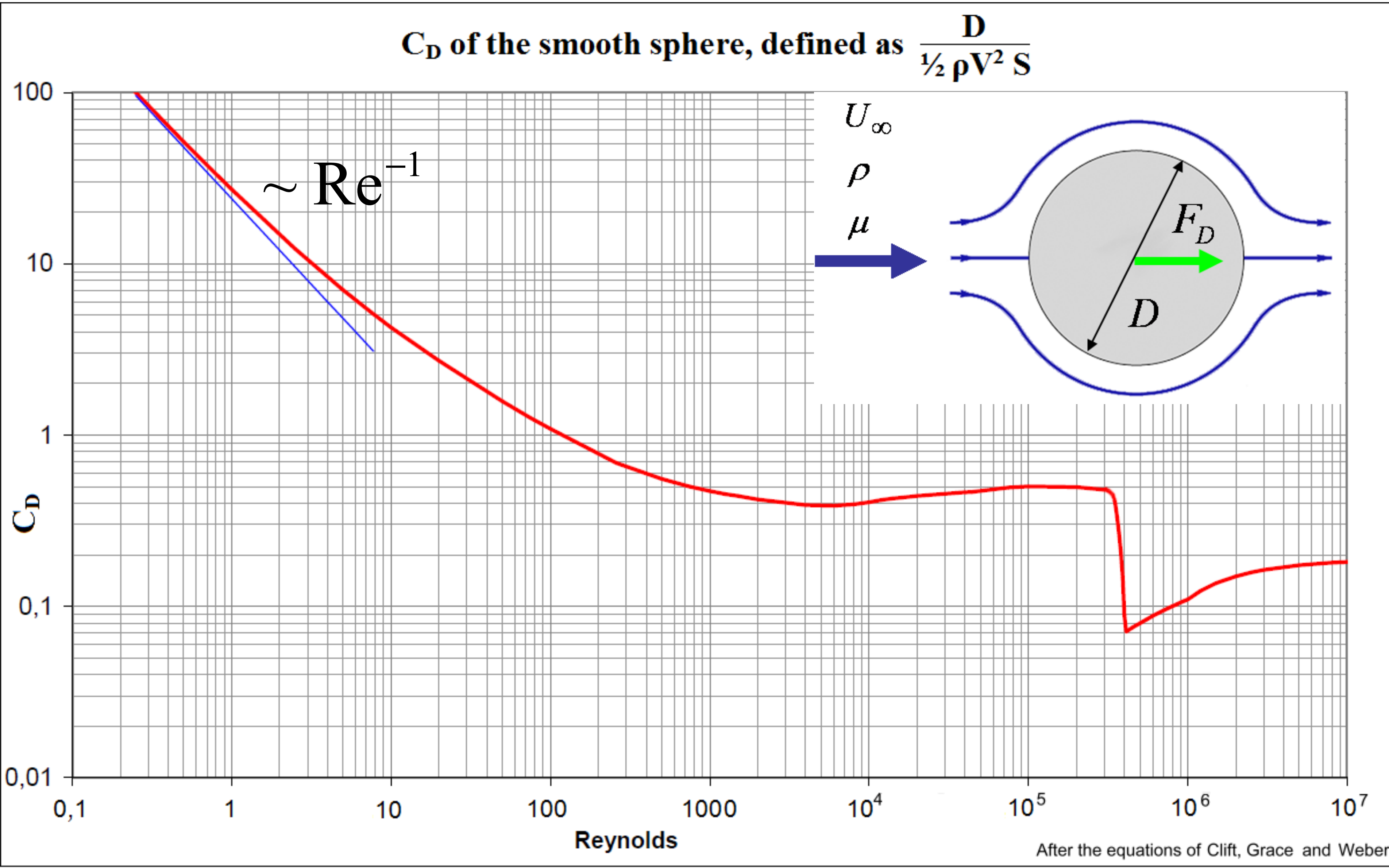
But what is this functional?

$$\frac{F}{U^2 D^2 \mu} = f(\mathrm{Re}^{-1})$$

https://www.youtube.com/watch?v=DkJt7668kFI

https://en.wikiversity.org/wiki/Fluid_Mechanics_for_Mechanical_Engineers/Dimensional_Analysis

$$\frac{F}{\rho U^2 D^2} = f\left(\mathrm{Re}^{-1}\right)$$



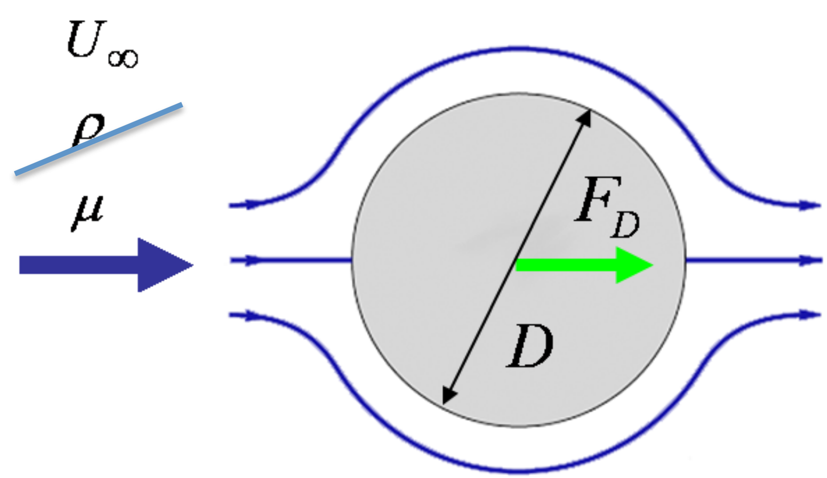
# What about colloids?

- Drag on a spherical colloid $\mathrm{Re} = \dfrac{\text{Inertial forces}}{\text{Viscous forces}} = \dfrac{\rho U L}{\mu} << 1$
  - Given $F_D = f(U, D, \mu, \rho)$
  - Objective: Find number of dimensionless groups

Parameters: $F_D, U, D, \mu$ i.e. p=4

Dimensions: Mass(M),Length(L),t i.e. d=3

Number of independent dimensionless groups, $\Pi$=p-d=4-3=1



$$\Pi_1 = F_D, U, D, \mu$$

Write the the parameters as a function of primary dimensions

$$F_D = [\frac{ML}{t^2}], U = [\frac{L}{t}], D = [L], \mu = [\frac{M}{Lt}]$$

$$\Pi_1 = \left[\frac{ML}{t^2}\right], \left[\frac{L}{t}\right]^{a1}, L^{b1}, \left[\frac{M}{Lt}\right]^{c1}$$

$$[M]: 1 + c1 = 0$$

$$[L]: 1 + a1 + b1 - c1 = 0$$

$$[t]: -2 - a1 - c1 = 0$$

c1=-1,a1=-1,b1=-1

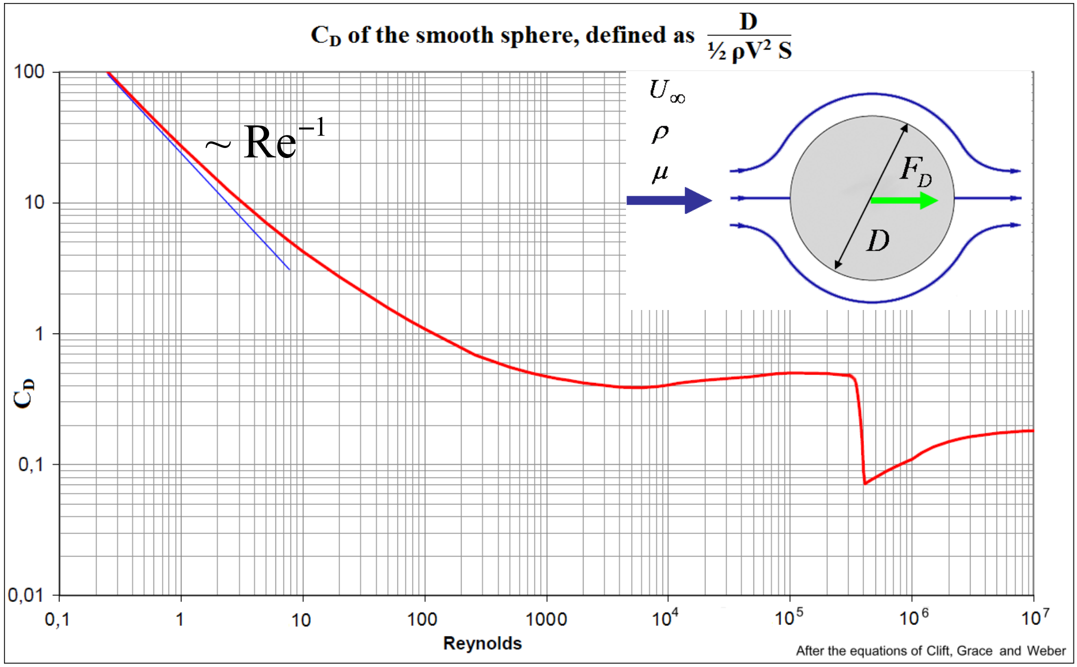
$$\Pi_1 = FU^{-1}D^{-1}\mu^{-1} = \frac{F}{UD\mu} = \text{constant}$$

# Experiment and Dimensional analysis together

$$FU^{-1}D^{-1}\mu^{-1} = \frac{F}{UD\mu} = C_1$$

$$\frac{F}{UD\mu}\frac{\mu}{\rho UD} = C_1\frac{\mu}{\rho UD}$$

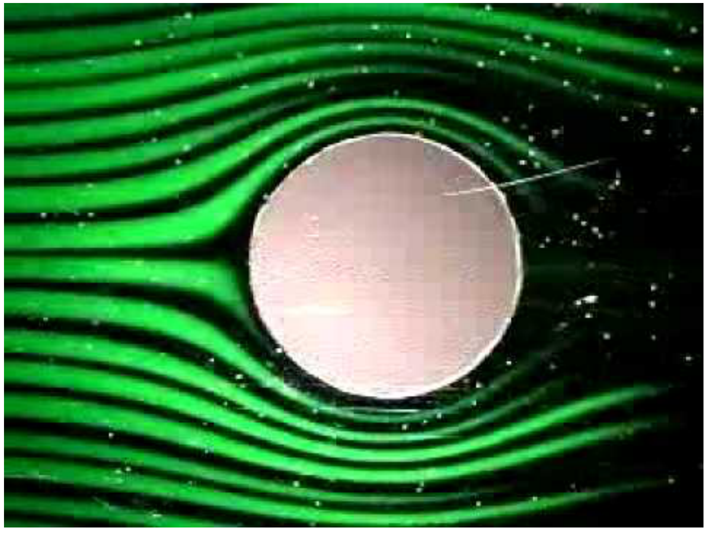
$$\frac{F}{\rho U^2 D^2} = \frac{C_1}{\text{Re}}$$




- I can calculate the drag on a sphere combining dimensional analysis and macroscopic experiments

# Introduction to Fluid Mechanics

- Describe the motion of fluids based on boundary conditions
  - Connect **u** (x,y,z,t) to P, stress while satisfying boundary conditions



Source: Wikipedia

# Introduction to Fluid Mechanics

- Notation P, **u** (u,v,w,t), $\underset{\sim}{\sigma}$
- Continuity $\frac{\partial \rho}{\partial t} + \Delta . \mathbf{u} = 0$ Eq.1

$\underset{\sim}{\sigma} = Tensor$

$\mathbf{u} = Vector$

- State of stress a.k.a Cauchy stress eqns.

State of stress $\rho \frac{Du}{\partial t} = \mathbf{f} + \nabla . \underset{\sim}{\sigma}$ Eq.2

$\mathbf{f} = \rho \mathbf{g}$

Navier-Stokes $\rho \frac{D\mathbf{u}}{\partial t} = -\nabla P_d + \mu \nabla^2 \mathbf{u}$ Eq.4

Consitutive law

$\underset{\sim}{\sigma} = -p\underset{\sim}{I} + 2\mu \underset{\sim}{E}$ Eq.3

$\underset{\sim}{\sigma}$ = stress tensor

$\underset{\sim}{E}$ = Rate of strain tensor

# Note about kinematics

- Kinematics describes fluid deformation of an arbitrary point in a velocity field

Local rotation

Rate of s train

$$\mathbf{u}(r) = \mathbf{u}(0) + \frac{1}{2}\mathbf{w} \wedge \mathbf{r} + \mathbf{r}.\underset{\sim}{E}$$

Vorticity $\mathbf{w} = \nabla \wedge \mathbf{u}$

Rate of strain $\underset{\sim}{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$

# From state of stress into Navier-Stokes

$$\rho \frac{Du}{\partial t} = \mathbf{f} + \nabla . \underset{\sim}{\sigma} \quad \text{Eq.2}$$

$$\rho \frac{Du}{\partial t} = \rho g - \nabla p + \mu \nabla^2 \mathbf{u}$$

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} \right] = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u}$$

$$P_d = p - \rho \mathbf{g}.\mathbf{r}$$

- Take the strain tensor (Eq.3) from constitutive law connecting stress to velocity field and viscosity substitute into Cauchy stress equation (Eq. 2)
- Assume incompressibility assumption

i.e. Viscosity constant over time.

- Assume gravity as the only body force you get Navier Stokes
- Assume Newtonian fluid

$$\rho \frac{D\mathbf{u}}{\partial t} = -\nabla P_d + \mu \nabla^2 \mathbf{u}$$

Eq.4 Navier-Stokes

$$\nabla . \mathbf{u} = 0$$

Eq.1 Continuity

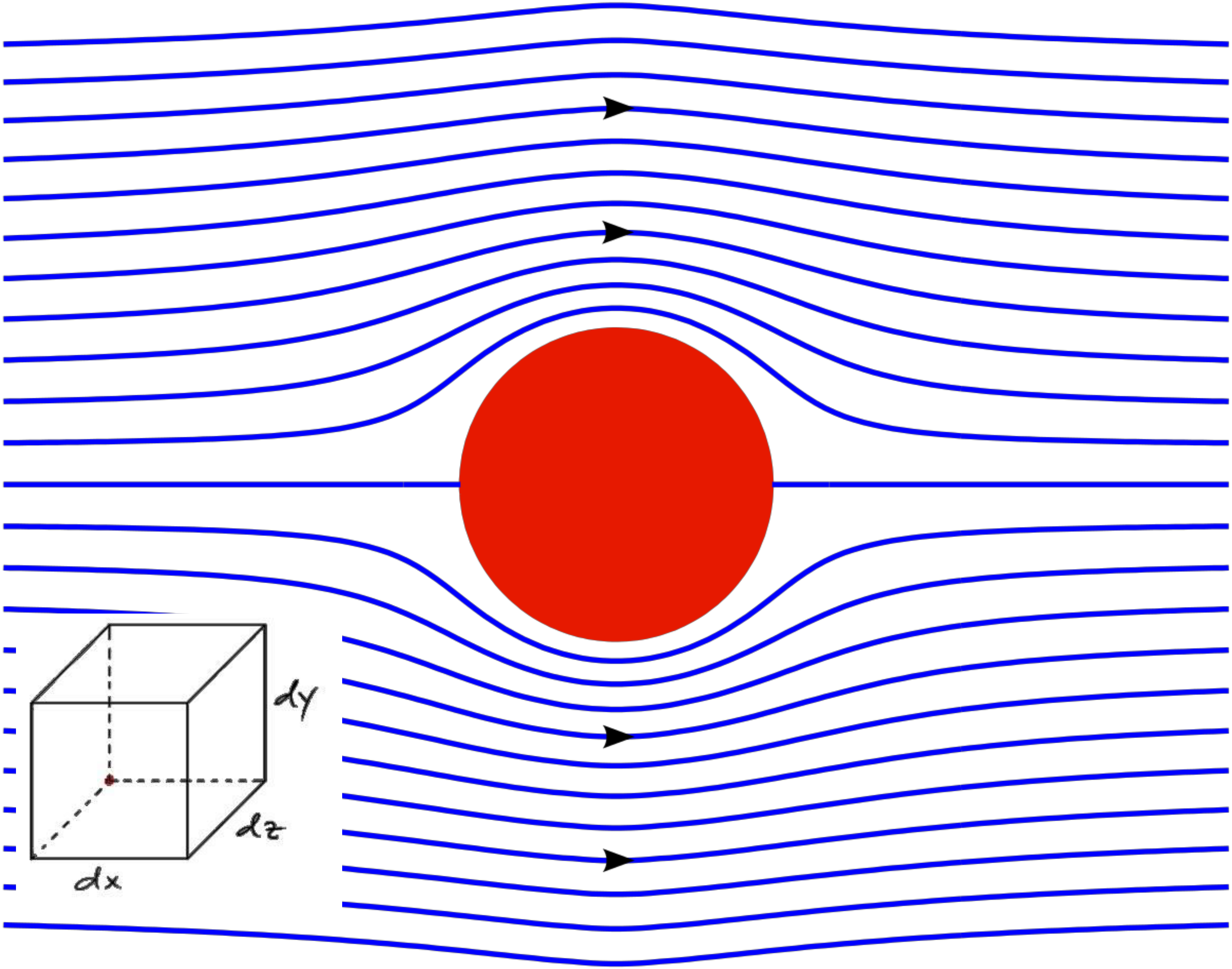
Physical understanding:
The pressure either overcomes
Viscous friction or accelerates fluid

# Derivation of continuity equation

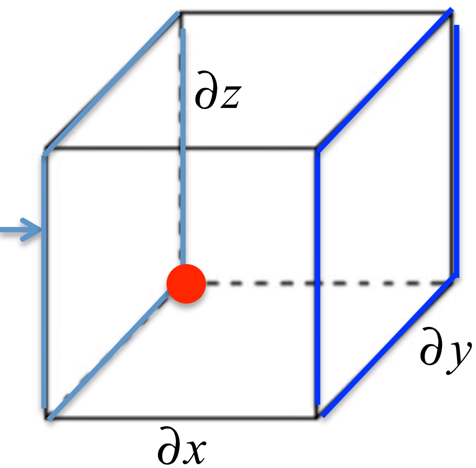
$$\frac{\partial \rho}{\partial t} + \Delta.\mathbf{u} = 0$$

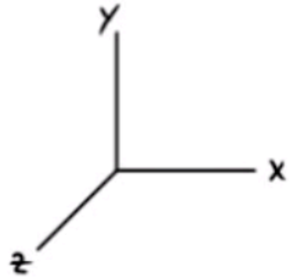
where u,v,w are velocities along x,y,z axises and $\rho$ is the density

$$\mathbf{u} = (u, v, w, t)$$




Fluid flowing through infinitesimally small control volume




$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial m}{dt}$$

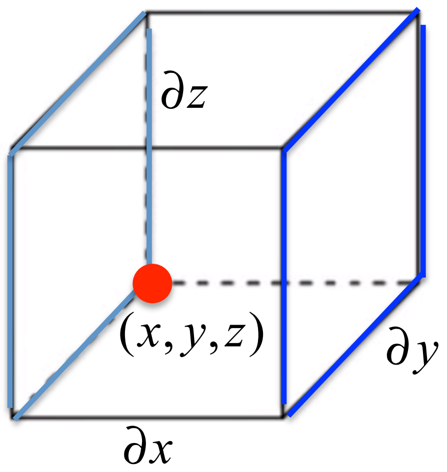
$$\mathbf{u} = (u, v, w, t)$$

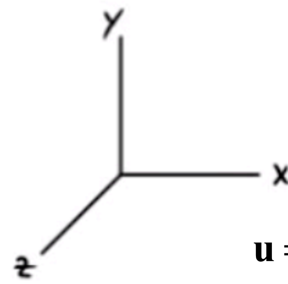
$$\dot{m}_{in} = \rho u \|_x \, \partial y \partial z + \rho v \|_y \, \partial x \partial z + \rho w \|_z \, \partial x \partial y$$

$$\dot{m}_{out} = \rho u \|_{x+\partial x} \, \partial y \partial z + \rho v \|_{y+\partial y} \, \partial x \partial z + \rho w \|_{z+\partial y} \, \partial x \partial y$$

$$m = \rho \partial x \partial y \partial z$$

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t} \partial x \partial y \partial z$$




$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial m}{dt} \quad \textbf{Eq.1}$$

$$\mathbf{u} = (u, v, w, t)$$

$$\dot{m}_{in} = \rho u \|_x \, \partial y \partial z + \rho v \|_y \, \partial x \partial z + \rho w \|_z \, \partial x \partial y \quad \textbf{Eq.2}$$

$$\dot{m}_{out} = \rho u \|_{x+\partial x} \, \partial y \partial z + \rho v \|_{y+\partial y} \, \partial x \partial z + \rho w \|_{z+\partial y} \, \partial x \partial y \quad \textbf{Eq.3}$$

$$m = \rho \partial x \partial y \partial z$$

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t} \partial x \partial y \partial z \quad \textbf{Eq.4}$$

Inserting equation 2,3,4 into equation 1

$$\rho u \|_x \, \partial y \partial z + \rho u \|_y \, \partial x \partial z + \rho u \|_z \, \partial x \partial y$$

$$-\rho u \|_{x+\partial x} \, \partial y \partial z - \rho u \|_{y+\partial y} \, \partial x \partial z - \rho u \|_{z+\partial y} \, \partial x \partial y = \frac{\partial \rho}{\partial t} \partial x \partial y \partial z$$

Divide both sides with dxdydz and rearrange

$$\frac{\rho u \|_{x+\partial x} - \rho u \|_x}{\partial x} + \frac{\rho u \|_{y+\partial y} - \rho u \|_y}{\partial y} + \frac{\rho u \|_{z+\partial y} - \rho u \|_z}{\partial z} = -\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \Delta.\mathbf{u} = 0$$

**If the flow is steady**

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

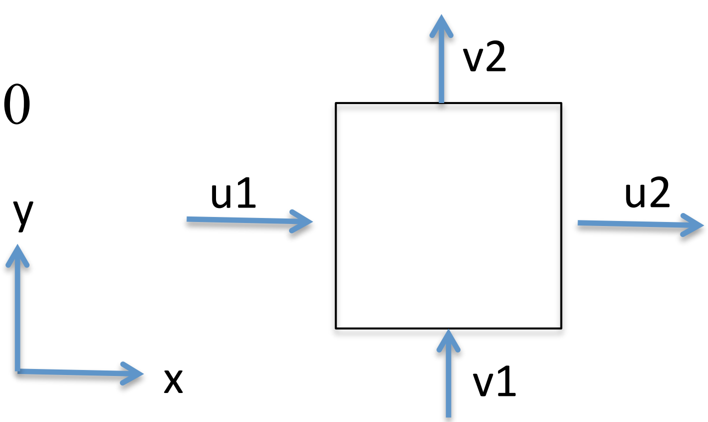
**If the flow is steady & incompressible**

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

**Physical understanding for steady & incompressible flow**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$




If u2>u1 then v1>v2

# Physical understanding from the continuity equation


$$\rho_2 A_2 v_2 = \rho_2 A_1 v_1$$

Same, incompressable, fluid so roe drops out!

$$A_1 v_1 = A_2 v_2$$

# Derivation of Navier-Stokes

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\frac{\sum F_x = \mathrm{ma}_\mathrm{x}}{volume}$$

$$\mathbf{u} = (u, v, w, t)$$

$$a_x = \frac{\partial u}{\partial t}$$

$$= \frac{\partial u}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial u}{\partial z}$$

Chain rule

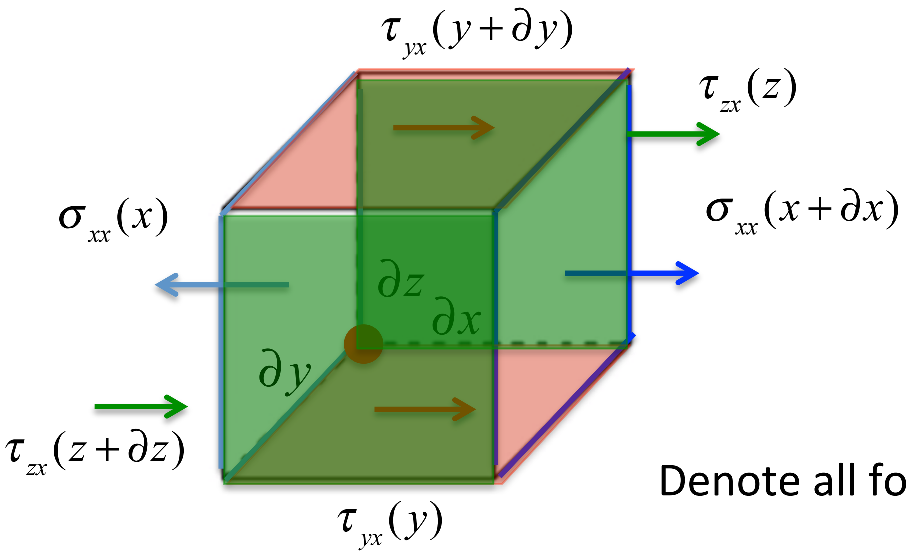
$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Result 1 on acceleration

Local acceleration

Convective acceleration

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$




$$\frac{\sum F_x = \mathrm{ma}_\mathrm{x}}{volume}$$

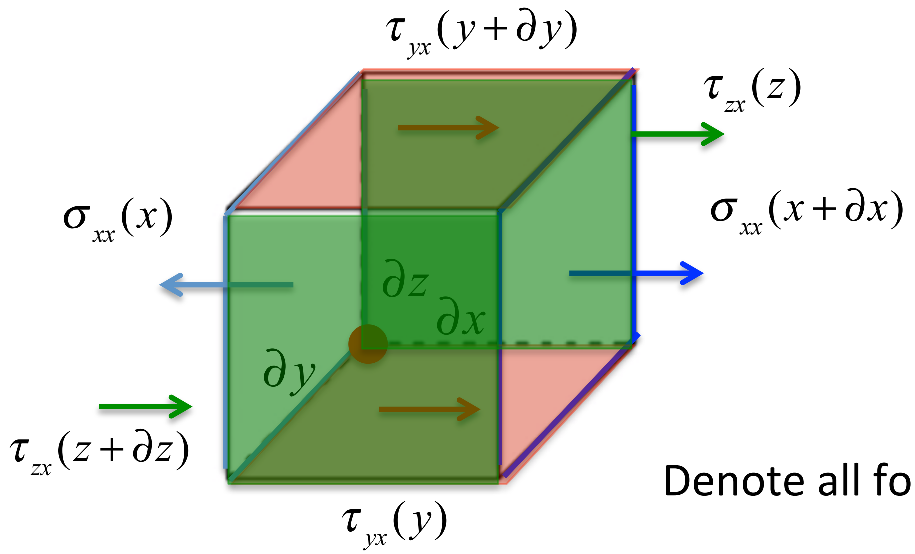
Rewrite body force in x direction

$$mg_x = \rho\partial x\partial y\partial z g_x$$

Denote all forces acting on x direction

$$\rho\partial x\partial y\partial z g_x + \sigma_{xx}(x+dx)\partial y\partial z - \sigma_{xx}(x)\partial y\partial z$$
$$+\tau_{yx}(y+\partial y)\partial x\partial z - \tau_{yx}(y)\partial x\partial z$$
$$+\tau_{zx}(z+\partial z)\partial x\partial z - \tau_{zx}(z)\partial x\partial z$$
$$= \rho\partial x\partial y\partial z a_x$$

$$\rho g_x - \frac{\partial P}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) = \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})$$




$$\frac{\sum F_x = \mathrm{ma}_x}{volume}$$

Rewrite body force in x direction

$$mg_x = \rho \partial x \partial y \partial z g_x$$

Denote all forces acting on x direction

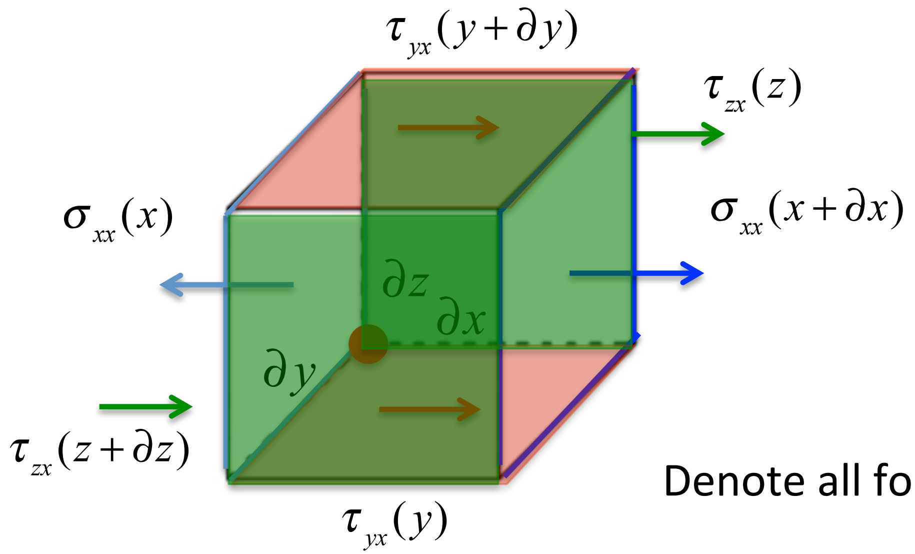
$$\frac{\rho \partial x \partial y \partial z g_x}{\partial x \partial y \partial z} + \frac{\sigma_{xx}(x+dx)\partial y \partial z - \sigma_{xx}(x)\partial y \partial z}{\partial x \partial y \partial z}$$

$$+\frac{\tau_{yx}(y+\partial y)\partial x \partial z - \tau_{yx}(y)\partial x \partial z}{\partial x \partial y \partial z}$$

$$+\frac{\tau_{zx}(z+\partial z)\partial x \partial z - \tau_{zx}(z)\partial x \partial z}{\partial x \partial y \partial z}$$

$$= \frac{\rho \partial x \partial y \partial z a_x}{\partial x \partial y \partial z}$$

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$




$$\frac{\sum F_x = \mathrm{ma}_x}{volume}$$

Rewrite body force in x direction

$$mg_x = \rho\,\partial x\,\partial y\,\partial z\,g_x$$

Denote all forces acting on x direction

$$\rho g_x + \frac{\sigma_{xx}(x+dx) - \sigma_{xx}(x)}{\partial x} + \frac{\tau_{yx}(y+\partial y) - \tau_{yx}(y)}{\partial y} + \frac{\tau_{zx}(z+\partial z) - \tau_{zx}(z)}{\partial z}$$

$$= \rho a_x$$

In the limit dx,dy,dz go to 0 i.e. infinitesimally small volume

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho a_x$$

**Replacing $a_x$ from slide 7 we get equation of motion for fluids**

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) \qquad \frac{\sum F_x = \text{ma}_x}{volume}$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) \qquad \frac{\sum F_y = \text{ma}_y}{volume}$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = \rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) \qquad \frac{\sum F_z = \text{ma}_z}{volume}$$

**We need expressions for normal and shear stresses i.e constitutive equations to connect stressed to fluid velocity. These equations are for an Newtonian fluid**

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} \qquad \tau_{xy} = \tau_{yx} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial x} \qquad \tau_{yz} = \tau_{yz} = \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial x} \qquad \tau_{zx} = \tau_{xz} = \mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$$

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})$$

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial x}$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial x}$$

$$\tau_{xy} = \tau_{yx} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

$$\tau_{yz} = \tau_{yz} = \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$$

$$\tau_{zx} = \tau_{xz} = \mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$$

$$\rho g_x + \frac{\partial(-p + 2\mu\frac{\partial u}{\partial x})}{\partial x} + \frac{\partial(\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}))}{\partial y} + \frac{\partial(\mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}))}{\partial z}$$

$$= \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})$$

$$\rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial(\frac{\partial v}{\partial x})}{\partial y} + \mu\frac{\partial(\frac{\partial w}{\partial x})}{\partial z} + \mu\frac{\partial^2 u}{\partial z^2}$$

$$= \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})$$

Write this as summation of two terms and rearrange

$$\rho g_x - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + +\mu \frac{\partial^2 u}{\partial z^2} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial (\frac{\partial v}{\partial x})}{\partial y} + \mu \frac{\partial (\frac{\partial w}{\partial x})}{\partial z}$$

$$= \rho (\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z})$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + \mu \frac{\partial (\frac{\partial u}{\partial x})}{\partial x} + \mu \frac{\partial (\frac{\partial v}{\partial x})}{\partial y} + \mu \frac{\partial (\frac{\partial w}{\partial x})}{\partial z}$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + \mu \frac{\partial}{\partial x} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

For incompressible Newtonian fluid

# Finally NS equation

We arrive at a NS equations for Newtonian incompressible fluid by moving LHS in * slide 13 to left

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Body force

Pressure gradient

Forces due to viscosity

Forces acceleration of fluid

$$\rho\frac{D\mathbf{u}}{\partial t} = -\nabla P_d + \mu\nabla^2\mathbf{u}$$

$$\nabla.\mathbf{u} = 0$$

# Navier-Stokes to Stokes at low Re

$$\rho \frac{D\mathbf{u}}{\partial t} = -\nabla P_d + \mu \nabla^2 \mathbf{u}$$

Navier-Stokes

$$\nabla . \mathbf{u} = 0$$

Continuity

$$\frac{\text{Inertial stress}}{\text{Viscous stress}} = \frac{O\left(\rho \frac{D\mathbf{u}}{\partial t}\right)}{O\left(\mu \nabla^2 \mathbf{u}\right)} = \frac{O\left(\rho u^2 / l\right)}{O\left(\mu u / l^2\right)} = \frac{\rho u l}{\mu} = \mathrm{Re}$$

# Inertial vs. viscous effects

$$P_d = p - \rho \mathbf{g}.\mathbf{r}$$

$$\rho \frac{D\mathbf{u}}{\partial t} = -\nabla P_d + \mu \nabla^2 \mathbf{u}$$ If Re<<1

$$\nabla.\mathbf{u} = 0$$

Stoke's Equation a.k.a Creeping flow

$$-\nabla P_d + \mu \nabla^2 \mathbf{u} = \mathbf{0}$$

$$\nabla.\mathbf{u} = 0$$

- **Time reversible** i.e. Stoke's equation is time independent or time invariant
- **Viscous dominated flow = No acceleration term** i.e. objects can not accelerate in low Re flows
- **Linear equation** i.e. additive

$$\mathrm{Re} = \frac{\rho u l}{\mu} = \frac{ul}{\upsilon} = \frac{l^2 / \upsilon}{l / u} = \frac{\text{Diffusion time scale of vorticity i.e.local rotation}}{\text{Advection time scale}}$$

$$\upsilon = \frac{\mu}{\rho} = [\frac{L^2}{t}] \Rightarrow \text{kinematic viscosity}$$

# Time reversibility

# Inertia versus Viscosity


M.Phelps 100 m butterfly 50.77 s

L = 2 m, U ≈ 2 m/s


Bellovibrio bacteriovorus

L = 2 $\mu$m, U = 100 $\mu$m/s

$$\text{Re} = \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{\rho U^2}{\frac{\mu U}{L}} = \frac{\rho U L}{\mu}$$

$\mu$=viscosity

U=speed of the object

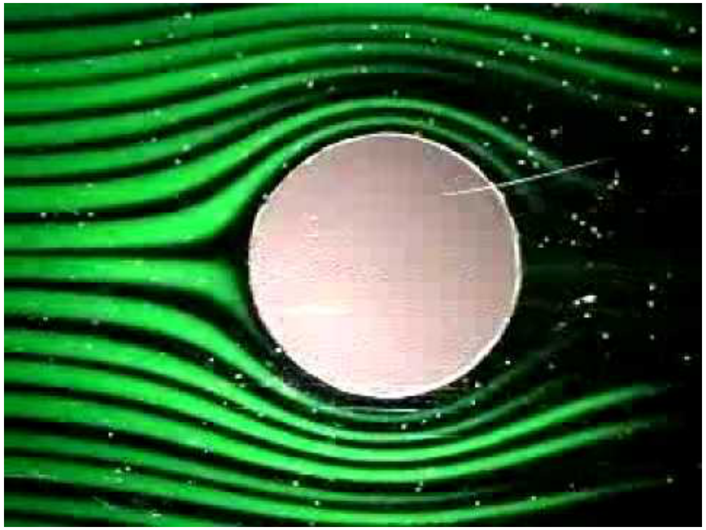
L=length of the object

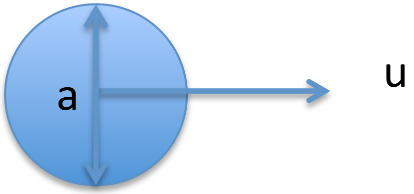
$\rho$=density of the object

**Guinness Book of World Records*. New York: Bantam. 1991: 128

G.I.Taylor, Cambridge University Press, NSF educational movies

# Flow past a sphere






**Scaling argument**

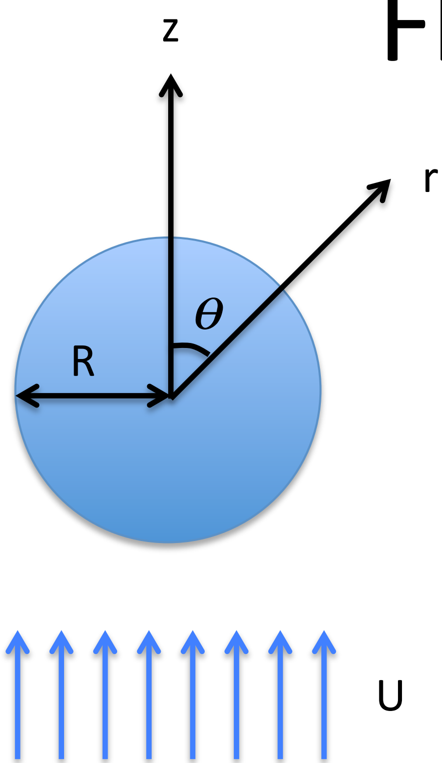
$$F = C_2 U a \mu$$

**Stoke's Equation**

$$F = 6\pi U a \mu$$

$$u(r) \propto \frac{1}{r}$$

# Flow past a sphere




$$Given: u_z = U \;\; z,r \to \infty, \mathrm{Re} << 1$$

$$Deduce: \text{System is rotationally symmetric around } \theta$$

$$Deduce: u_\theta = 0$$

Goal: Determine uz(r,$\theta$), ur(r,$\theta$), P(r,$\theta$)

# Flow past a sphere

$e_z = \cos\theta e_r - \sin\theta e_\theta$ where $e_z, e_r, e_\theta$ are unit vectors in spherical coordinates

$\mathrm{u}(\infty,\theta)=\mathrm{U}e_z = \mathrm{U}\cos\theta e_r - \mathrm{U}\sin\theta e_\theta$

$\mathrm{u}_\mathrm{r} = U\cos\theta, \mathrm{u}_\theta = -U\sin\theta \ \mathrm{r} \to \infty$

Assume all velocity components are separable later check !!!

Boundary conditions:

$BC1: \mathrm{u}_\mathrm{r}(r,\theta) = f(r)\cos\theta, f(r) = 0, f(\infty) = U$

$BC2: \mathrm{u}_\theta(r,\theta) = g(r)\sin\theta, g(r) = 0, g(\infty) = -U$

$BC3:$ No slip@r=R,g(R)=0

$BC4:$ No penetration@r=R,f(R)=0

Deduce that P mimics $\mathrm{u}_\mathrm{r}$

$\mathrm{P}(\mathrm{r},\theta)=\mathrm{h}(\mathrm{r})\cos\theta$

W.M.Deen Chapter 8.3

# Continuity in spherical coordinate

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f\cos\theta) = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(g\sin\theta\sin\theta) = -\frac{2g}{r}\cos\theta$$

Eliminate the common factor $\cos\theta$ and rearrange

$$g = -\frac{r}{2}\frac{\partial f}{\partial r} - f$$

Continuity equation

# Axisymmetric Stokes equations in spherical coordinate

$$\frac{1}{\mu}\frac{\partial}{\partial r}P = \nabla^2 u_r - \frac{2}{r^2}(u_r + \frac{\partial u_\theta}{\partial \theta} + u_\theta \cot\theta)$$

$$\frac{1}{\mu r}\frac{\partial}{\partial \theta}P = \nabla^2 u_\theta + \frac{2}{r^2}(\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{2\sin^2\theta})$$

Stoke's equations

$$\frac{1}{\mu}\frac{\partial h}{\partial r} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) - 4(f+g)\right]$$

$$\frac{h}{\mu} = -\frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\frac{\partial g}{\partial r}) - 2(f+g)\right]$$

Continuity equation

$$g = -\frac{r}{2}\frac{\partial f}{\partial r} - f$$

Three linear differential equations
Three unknown function of r !!!

# Flow past a sphere

- Eliminate h,g to arrive at a Euler equation for f with known general solution

$$\frac{df^4}{dr^4}+\frac{8d^3 f}{dr^3}+\frac{8d^2 f}{r^2 dr^2}-\frac{8d^3 f}{r^3 dr^3}=0$$

$$f(r)=Ar^2+B+\frac{C}{r}+\frac{D}{r^3}$$ Known general equation

$$g(r)=-B-\frac{C}{2r}+\frac{D}{r^3}$$ Replace continuity to solve

$$\frac{h(r)}{\mu}=2Ar+\frac{C}{r^2}$$ Replace in Stoke's to solve

Solving for boundary conditions:

$$A = 0, B = U, C = -\frac{3}{2}UR, D = \frac{UR^3}{2}$$

Substiting A,B,C,D gives

$$\frac{u_r(r,\theta)}{U} = \left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{3}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta$$

$$\frac{u_\theta(r,\theta)}{U} = \left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta$$

$$\frac{P(r,\theta)}{\mu U / R} = -\frac{3}{2}\left(\frac{R}{r}\right)^2 \cos\theta$$

- Absence of inertia→Pressure disturbances are comparable to viscous stress
- P increases upstream, decrease downstream
- P variations contribute to drag and shear stress at the surface of the sphere.

# Drag on a sphere

Force fluid excerts on a surface

$$\mathrm{F}=\int_S \mathbf{n}.\underline{\underline{\sigma}}\,\partial S$$

*Generalized form for any axisymmetric flow past a solid sphere*

$$F_D = -2\pi R^2 \int_0^{\pi} [P(r,\theta]\cos\theta + \tau_{r\theta}(r,\theta)\sin\theta]\sin\theta\,\partial\theta$$

Torque a fluid excerts on a body

$$\mathrm{L}^{\mathrm{H}} = \int_S \mathbf{r} \wedge (\mathbf{n}.\underline{\underline{\sigma}})$$

$$F_D = 3\pi\mu UR \int_0^{\pi} [\cos^2\theta + \sin^2\theta]\sin\theta\,\partial\theta$$

Using the solution of flow past a sphere

$$\int_0^{\pi} \cos^2\theta\sin\theta\,\partial\theta = \frac{2}{3}, \int_0^{\pi} \sin^2\theta\sin\theta\,\partial\theta = \frac{4}{3}$$

$$F_D = 6\pi\mu UR$$

$$\frac{P(r,\theta)}{\mu U / R} = -\frac{3}{2}\left(\frac{R}{r}\right)^2 \cos\theta$$

$$\tau_{r\theta} = -\frac{3}{2}\left(\frac{\mu U}{r}\right)\sin\theta$$

# Diffusion of colloids: Einstein-Sutherland-Stokes

$$\text{Diffusion coefficient}, D = \frac{kT}{\frac{F^H}{u}} = \frac{\text{Kinetic energy of surrounding molecules}}{\text{Resistence of the fluid}}$$

$$\text{For spheres } \mathrm{F}^{\mathrm{H}} = 6\pi\mu u a$$

$$D = \frac{kT}{6\pi\mu a}$$

# Supplementary information

# Force and Torque a Fluid exerts on a body

$$\nabla.\mathbf{u} = 0 \text{ and } \nabla.\underline{\underline{\sigma}} = 0$$

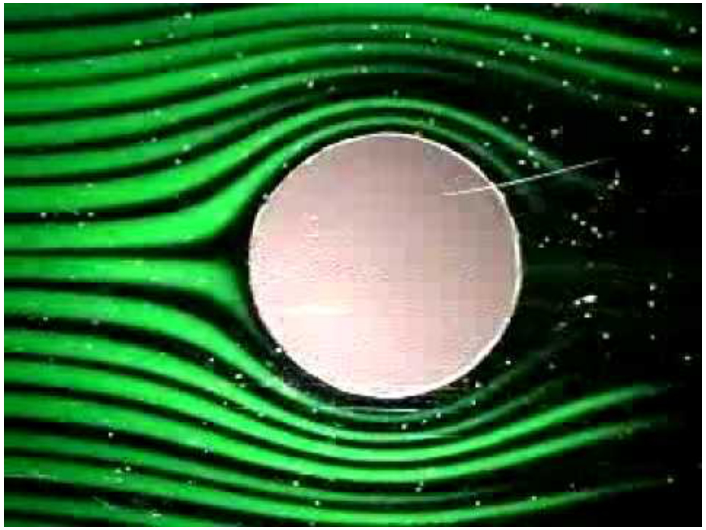
$$\underline{\underline{\sigma}} = -p_d \underline{\underline{I}} + 2\mu \underline{\underline{E}}$$

Force fluid excerts on a surface

$$\text{F}=\int_S \mathbf{n}.\underline{\underline{\sigma}}\, \partial S$$

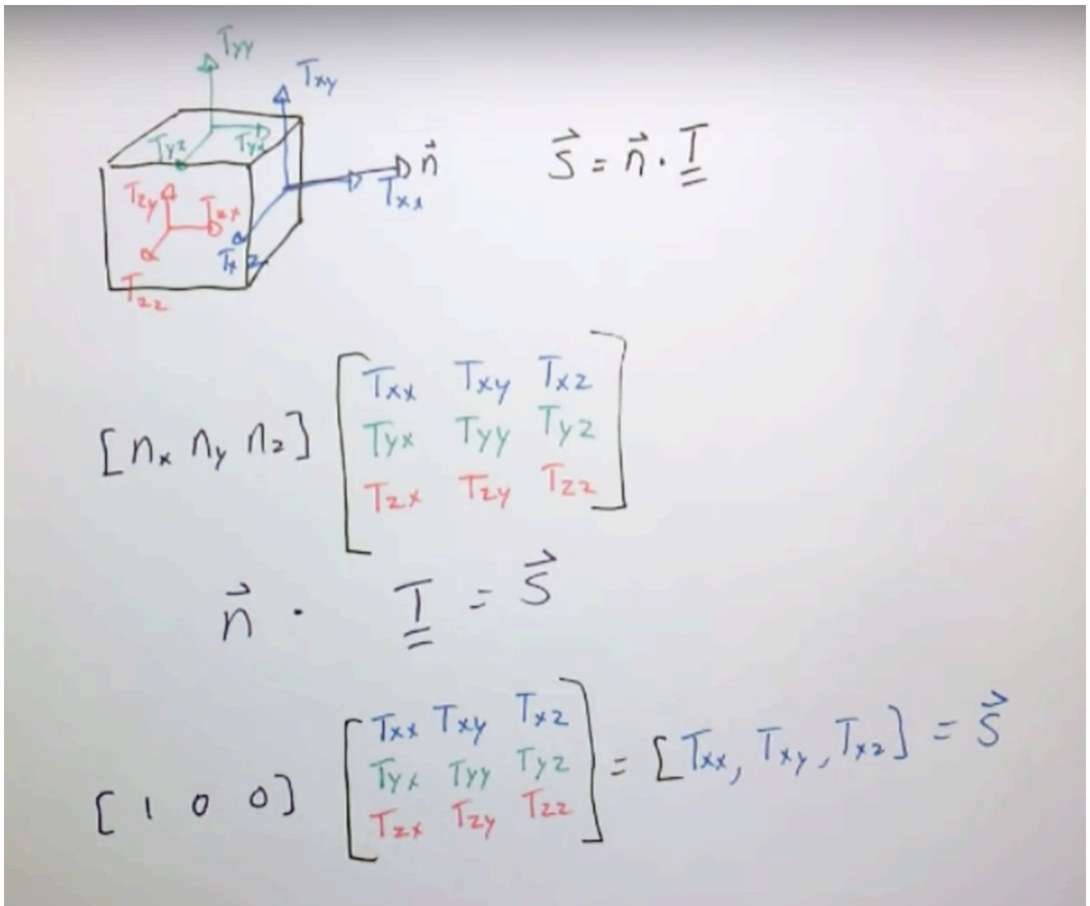
Torque a fluid excerts on a body

$$\text{L}^{\text{H}} = \int_S \mathbf{r} \wedge (\mathbf{n}.\underline{\underline{\sigma}})$$



# What is a stress tensor?

https://www.youtube.com/watch?v=uO_bW2zzrNU



# Vector calculus review

- Vector calculus review link
- Vector calculus review with Divergence theorem link